C. Analysis

Cauchy theorem

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Th 1. If $f(z)$ is analytic function of z and if $f(z)$ is continuous at each point within and on a closed contour C, then

$$\int_C f(z)\,dz = 0$$

Proof: In proving the above result we shall make use of the two results a
Green Theorem for a plane and Cauchy Riemman Equations.

(1.) If $P(x,y)$, $Q(x,y)$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ are all continuous function within a domain D and if C is any closed contour in D, then Green's theorem states that

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \times dx\,dy \quad \text{--- (1)}$$

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2.) Cauchy - Riemann Equations.

If $f(z) = u(x,y) + iv(x,y)$ be an analytic function then

$$\frac{du}{dx} = \frac{\partial v}{\partial x} \text{ and } \frac{\partial u}{dy} = \frac{-\partial v}{\partial x}$$

or $u_x = v_y$ & $u_y = -v_x$ — (2)

Now,

$f(z) = u + iv$ is analytic and has a continuous derivative.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

by (2)

From above it follows that u, v and the partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are all continuous inside and on C. Thus the Green's theorem can be applied.

$$\int_C f(z)\,dz = \int (u+iv)(dx+idy)$$

$$= \int (u\,dx - v\,dy) + i\,v\,dx + u\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx.dy$$

by (1)

$$= -\iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx.dy$$

$$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx.dy.$$

by (1)

$$= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

$$+ i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx.dy$$

$$= 0 + i\,0$$ by (2)

Hence $\int_C f(z)\,dz = 0$